Point form of Circle and Ellipse

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The idea of point circle and point ellipse and some of their applications.

Part 2
Point Ellipse

- **Point Ellipse:**

Point Ellipse is a point form of the closed locus named ellipse[1]. To understand the idea of point ellipse it will be convenient to look in to its equation.

We know the equation of ellipse having center at (h,k) and major axis $2p$ [parallel to x axis], minor axis $2q$ [parallel to y axis] is given by

$$\frac{(x-h)^2}{p^2}+\frac{(y-k)^2}{q^2}=1 \ldots. \quad \ldots. \quad \ldots. \ (1)$$

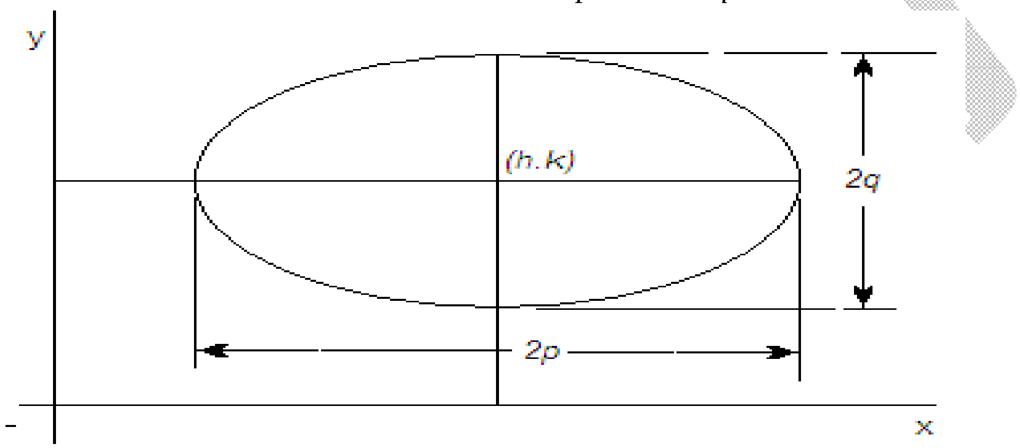

Fig: 1

Let $p = mr$ & $q = nr$ where [m, n, r are positive numbers and $m > n$]
Then from (1) we obtain

$$\frac{(x-h)^2}{m^2}+\frac{(y-k)^2}{n^2}=r^2 \ \ldots.. \ \ldots\ldots. \ \ldots\ldots. \ (2)$$

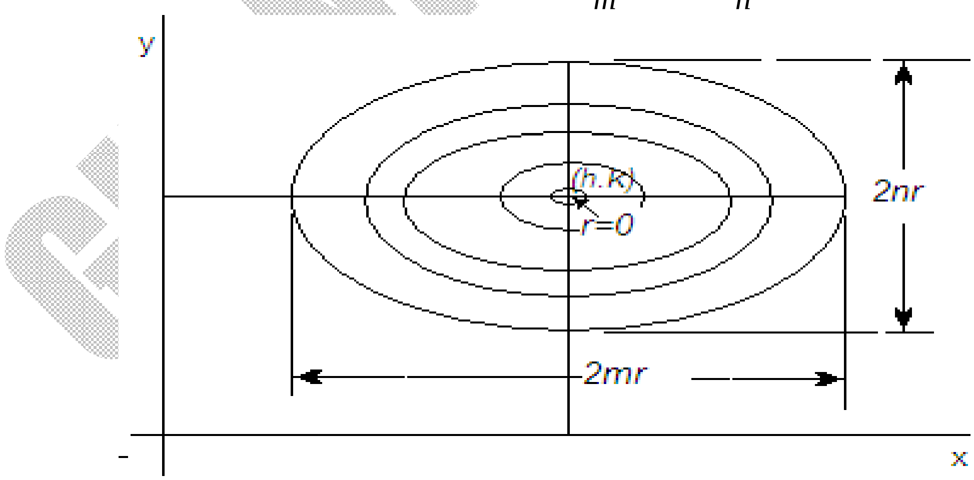

Fig:2

If m & n are constants and the value of r gets smaller then the ellipse becomes smaller keeping its center at the same point (h,k).

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[1] A catalog of special plane curves by J. Dennis Lawrence, Publisher: Dover Publications, 1972

For $r = 0$ the equation reduces to

$$\frac{(x-h)^2}{m^2} + \frac{(y-k)^2}{n^2} = 0 \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots(3)$$

Equation (3) is satisfied by one and only real point (h,k) and also its form is similar to that of an ellipse of eccentricity $e = \sqrt{1-\frac{n^2}{m^2}} = \sqrt{1-\frac{q^2}{p^2}}$. So we have reasons to call the equation as "Point Ellipse". "Point" is for it is satisfied by a single point and "Ellipse" is for it is similar to the shape of ellipse as the equation describes.

We can write a more general expression for point ellipse at a given point (h,k) as follows,

$$a^2(x-h)^2 + b^2(y-k)^2 = 0 \quad \ldots. \quad \ldots. \quad \ldots. \quad \ldots. \quad \ldots.(4)$$

$$[\,a,b \in R \;\&\; a,b \neq 0 \;\&\; a \neq b\,]$$

- **Application of Point Ellipse:**

Point ellipse is a point where all the points of an ellipse are assumed to be coexisting. So, for a given curve if we derive an equation of curves which are passing through the point of intersection of the given curve and a point ellipse (on the curve) then eventually the derived equation represents the family of touching curves. Let we have a given point (*h,k*) on curve L_1. Now the family of loci L_2 (curves) to touch the curve L_1 through point (*h,k*) is given by

$$L_2 \equiv L_1 + \lambda\{a^2(x-h)^2 + b^2(y-k)^2\} = 0 \;\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad (5)$$

[λ Is an arbitrary constant]

Now we will discuss some special cases applicable for college level analytic geometry.

- **Application 1:**

For $L_1 \equiv \frac{X^2}{p^2} + \frac{Y^2}{q^2} - 1 = 0$ & $\lambda = -1$; if a^2& b^2 is so chosen that equation (5) gets free from 2nd order terms of x & y then L_2 will be the tangent to the ellipse.

Example 1.1:

Find the equation of the tangent touching the ellipse $\frac{x^2}{16} + \frac{y^2}{12} = 1$ at point (2, 3).

Solution:

The equation of the tangent is given by

$$\frac{x^2}{16} + \frac{y^2}{12} - 1 - \left(\frac{(x-2)^2}{16} + \frac{(y-3)^2}{12}\right) = 0$$

$$\Rightarrow \frac{4x}{16}+\frac{6y}{12}-\frac{4}{16}-\frac{9}{12}-1=0$$
$$\Rightarrow x+2y-8=0 \ldots Ans$$

Example 1.2:

Find the equation of tangent touching the ellipse $\frac{(2x+y-6)^2}{9}+\frac{(x-2y+3)^2}{4}-\frac{4}{9}=0$; at the point (1,2)

Solution:

We can write an equation of point ellipse at point (1, 2) as,

$$\frac{\{2(x-1)+(y-2)\}^2}{9}+\frac{\{(x-1)-2(y-2)\}^2}{4}=0$$
$$\Rightarrow \frac{(2x+y-4)^2}{9}+\frac{(x-2y+3)^2}{4}=0 \ldots \quad \ldots \quad \ldots \quad \ldots (6)$$

And the given ellipse is

$$\frac{(2x+y-6)^2}{9}+\frac{(x-2y+3)^2}{4}-\frac{4}{9}=0 \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (7)$$

Now subtracting (6) from (7) giving the tangent

$$\frac{1}{9}\{(4x+2y-10)(-2)\}+\frac{1}{4}\{(2x-4y+6)(0)\}-\frac{4}{9}=0$$
$$\Rightarrow -2(4x+2y-10)-4=0$$
$$\Rightarrow 2x+y-4=0 \ Ans.$$

- **Application 2:**

We can easily choose the values of a^2 & b^2 so as to make the form of equation similar to that of circle.

Example 2.1: Find the equation of a circle touching the ellipse $\frac{x^2}{16}+\frac{y^2}{12}=1$ at the point (2,3) and passing through another point (7,0)

Solution:

Let the Equation of a circle touching the given ellipse at point (2,3) be,

$$\frac{x^2}{16}+\frac{y^2}{12}-1+\left(a(x-2)^2+b(y-3)^2\right)=0\,; \ldots \quad \ldots \quad \ldots \quad (8)$$

[Here *a* & *b* are arbitrary constants and *ab>0*]

For the condition of (8) being the equation of circle,

$$a+\frac{1}{16}=b+\frac{1}{12} \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (9)$$

Again for satisfying the circle by the point (0,5)

$$\frac{0^2}{16}+\frac{5^2}{12}-1+\left(a(0-2)^2+b(5-3)^2\right)=0$$

$$\Rightarrow \frac{25}{12}-1+4a+4b=0$$

$$\Rightarrow a+b=-\frac{13}{48} \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots (10)$$

Solving equation (9) & (10)

$$a=-\frac{1}{8}\ ;\ b=-\frac{7}{48}$$

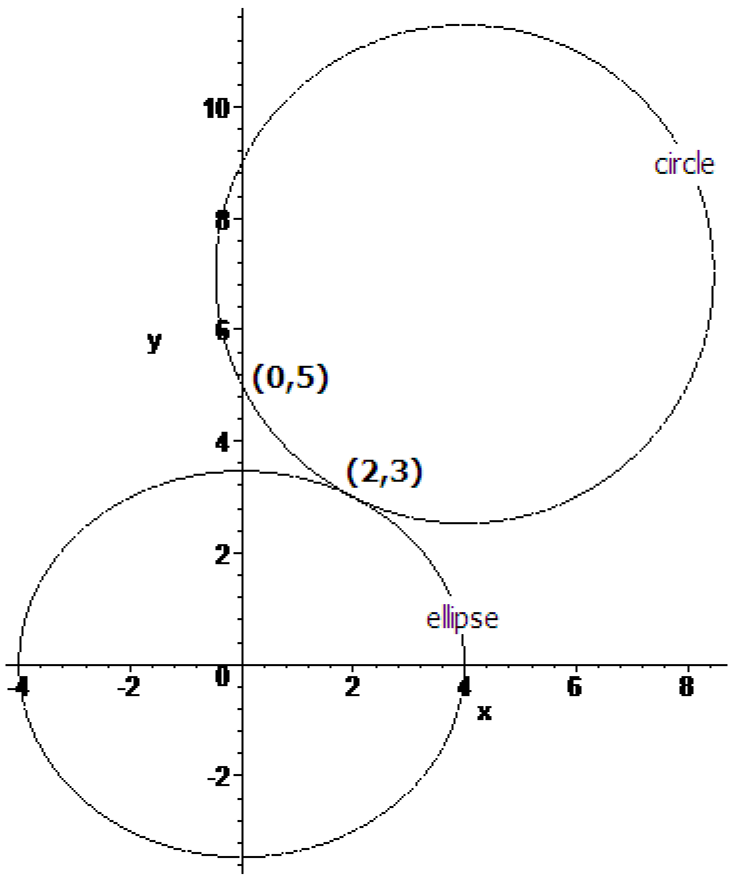

Fig: derived circle touching the given ellipse from equation(8)

Now put the value of a & *b* in (8) and the equation of the circle will be

$$x^2+y^2-8x-14y+45=0 \quad \text{Ans.}$$

Example 2.2: Find the equation of a conic touching the straight line $2x-5y-1=0$ at point (3,1)

Solution:
We can write the equation of the family of conics as follows

$$a(x-3)^2+b(y-1)^2+\lambda(2x-5y-1)=0 \ldots.. \quad \ldots.. \quad \ldots.. \quad \ldots.. \quad \ldots..(11)$$

[Here λ is a non zero arbitrary constant and $a\times b>0$]

Above equation can be applied to have three arbitrary parameters dependent touching curves keeping the tangency fixed at the same point of the given straight line. Following figure shows the real graph of equation (11) for different values of a, b & λ.

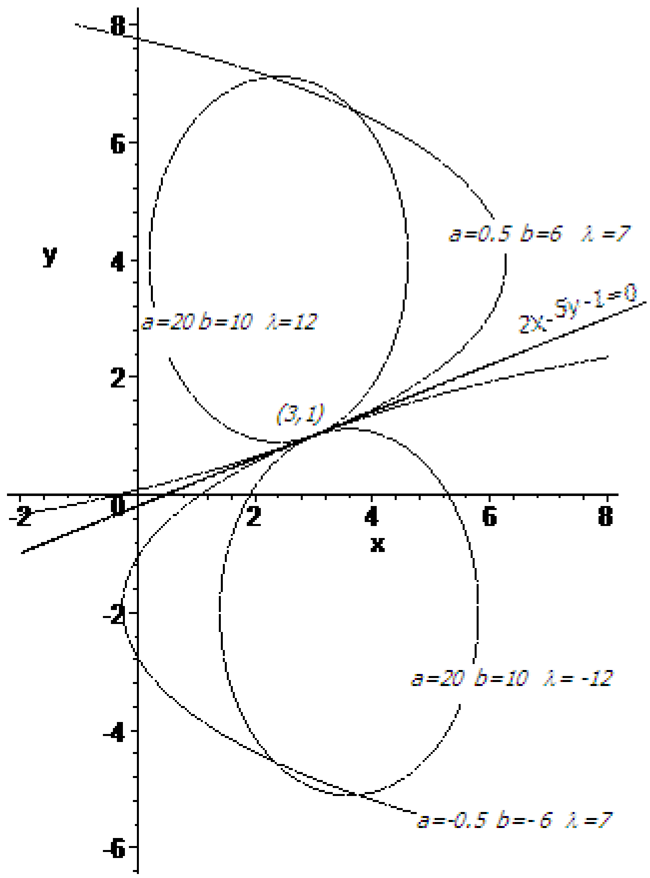

Fig: Family of conics touching a straight line at point(3,1)

Obviously all the curves obtained are conic sections. We plotted the graph using Maple 6.

All other above mentioned examples were tested by plotting the real graph using Maple 6 and found required property of the tangency.
